Mathematical Models of Smart Obstacles

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ABSTRACT

We propose mathematical models to describe the behaviour of smart obstacles. In the context of acoustic scattering a smart obstacle in an obstacle that when hit b y an incoming acoustic wave reacts circulating on its boundary a "pressure current" to pursue a given goal.

A pressure current is a quantity whose physical dimension is pressure divided by time. The goals considered are: 1) to be undetectable, 2) to appear with a shape and an acoustic boundary impedance different from its actual ones, 3) to appear in a location in space different from its actual one eventually with a shape and an acoustic boundary impedance different from its actual ones. He mathematical models proposed for the smart obstacles are optimal control problem for the wave equation. These optimal control problems are studied analytically and solved quantitatively using ad hoc numerical methods.

1. 0 INTRODUCTION

In this paper a mathematical model for an acoustic time dependent scattering problem involving smart obstacles is formulated. Smart obstacles are obstacles that when hit by an incoming acoustic field react in order to pursue an assigned goal. The goal pursued by the smart obstacle considered in this paper is: to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones. We call this goal: to appear as a ghost obstacle. The smart obstacle pursues its goal circulating a pressure current (i.e. a quantity whose physical dimension is pressure divided by time) on its boundary. We show that the pressure current necessary to pursue the goal can be determined as the solution of a suitable optimal control problem for the wave equation.

The author and its coworkers have studied similar models for several other classes of smart obstacles in acoustic and electromagnetic scattering (see for example (1)-(6) and the website: http://www.econ.univpm.it/recchioni). The obstacles considered pursue one of the following goals:

1. to be undetectable (i.e.: furtivity problem),
2. to appear with a shape and a boundary impedance different from its actual shape and impedance (i.e.: masking problem),
3. to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones (i.e.: ghost obstacle problem).

Zirilli, F. (2006) Mathematical Models of Smart Obstacles. In *Battlefield Acoustic Sensing for ISR Applications* (pp. 20-1 – 20-8). Meeting Proceedings RTO-MP-SET-107, Paper 20. Neuilly-sur-Seine, France: RTO. Available from: http://www.rto.nato.int/abstracts.asp.

The scattering problems corresponding to 1.-3. have been formulated as optimal control problems for the wave equation (acoustic case) or for the Maxwell equations (electromagnetic case) and the first order optimality conditions for these control problems have been derived applying the Pontryagin maximum principle and solved with appropriate numerical methods on several test problems. The choice of limiting the exposition to the ghost obstacle problem is motivated by the following reasons = necessity to choose a problem to fix the ideas and brevity and the fact that the ghost obstacle problem is relevant in applications and is harder than the furtivity and the masking problems. Several other approaches to study smart obstacles have been considered in the literature, see for example (7)-(10).

2.0 THE GHOST OBSTACLE OPTIMAL CONTROL PROBLEM

Let $\Omega \subset \mathbf{R}^3, \Omega_G \subset \mathbf{R}^3$ be two bounded simply connected open sets with locally Lipschitz boundaries $\partial\Omega, \partial\Omega_G$ and let $\overline{\Omega}$ and $\overline{\Omega}_G$ be their closures respectively. Let us denote with $\underline{n}(\underline{x}) = (n_1(\underline{x}), n_2(\underline{x}), n_3(\underline{x}))^T \in \mathbf{R}^3, \underline{x} \in \partial\Omega$ the outward unit normal vector to $\partial\Omega$ in $\underline{x} \in \partial\Omega$. Since Ω has a locally Lipschitz boundary, $\underline{n}(\underline{x}), \underline{x} \in \partial\Omega$, exists almost everywhere, similar statements hold for the outward unit normal vector to $\partial\Omega_G$. Furthermore let Ω_G be such that $\Omega_G \neq \emptyset$ and $\overline{\Omega} \cap \overline{\Omega}_G = \emptyset$. We assume that Ω and Ω_G are characterized by constant acoustic boundary impedances $\chi \geq 0$ and $\chi_G \geq 0$, respectively. The case $\chi = +\infty$ and/or $\chi_G = +\infty$ (i.e.: the case of acoustically hard obstacles) can be treated with simple modifications of the formulae presented here. We refer to $(\Omega; \chi)$ as the obstacle and to $(\Omega_G; \chi_G)$ as the ghost obstacle. We consider an acoustic incident wave $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$, propagating in a homogeneous isotropic medium in equilibrium at rest with no source terms present that satisfies the wave equation with wave propagation velocity $c > 0$ in $\mathbf{R}^3 \times \mathbf{R}$.
Finally we denote with $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ and with $u_G^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}_G) \times \mathbf{R}$, the waves scattered respectively by the obstacle $(\Omega; \chi)$ and by the ghost obstacle $(\Omega_G; \chi_G)$ when hit by $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$.
The scattered acoustic field $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ is defined as the solution of the following exterior problem for the wave equation:

$$\Delta u^s(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2 u^s}{\partial t^2}(\underline{x},t) = 0, \quad (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \tag{1}$$

with the boundary condition:

$$-\frac{\partial u^s}{\partial t}(\underline{x},t) + c\chi \frac{\partial u^s}{\partial \underline{n}(\underline{x})}(\underline{x},t) = g(\underline{x},t), \quad (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \tag{2}$$

where $g(\underline{x},t)$ is given by:

$$g(\underline{x},t) = \frac{\partial u^i}{\partial t}(\underline{x},t) - c\chi \frac{\partial u^i}{\partial \underline{n}(\underline{x})}(\underline{x},t), \quad (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \tag{3}$$

the boundary condition at infinity:

$$u^s(\underline{x},t) = O\left(\frac{1}{r}\right), r \to +\infty, \quad t \in \mathbf{R}, \tag{4}$$

and the radiation condition:

$$\frac{\partial u^s}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial u^s}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty, t\in\mathbf{R}, \tag{5}$$

where $r=\|\underline{x}\|$, $\underline{x}\in R^3, \Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $c>0$ is the wave propagation velocity and $O(\cdot)$ and $o(\cdot)$ are the Landau symbols. We note that $g(\underline{x},t)$, $(\underline{x},t)\in\partial\Omega\times\mathbf{R}$ is defined almost everywhere and that the boundary condition (2) can be adapted to deal with the limit case of the acoustically hard obstacles, i.e. $\chi=+\infty$. The obstacle $(\Omega;\chi)$ that scatters the field u^s solution of (1), (2), (3), (4), (5) is called passive obstacle. The field $u_G^s(\underline{x},t), (\underline{x},t)\in\left(\mathbf{R}^3\setminus\overline{\Omega}_G\right)\times\mathbf{R}$ scattered by the (passive) ghost obstacle is defined as the solution of (1), (2), (3), (4), (5) when in the problem defined above we replace Ω with Ω_G and χ with χ_G. Note that we always consider the ghost obstacle as a passive obstacle.

We consider the following problem:

Ghost Obstacle Problem: Given an incoming acoustic field $u^i(\underline{x},t), (\underline{x},t)\in\mathbf{R}^3\times\mathbf{R},$ an obstacle $(\Omega;\chi)$, a ghost obstacle $(\Omega_G;\chi_G)$ choose a pressure current circulating on $\partial\Omega$ for $t\in\mathbf{R}$ in such a way that the wave scattered by $(\Omega;\chi)$ when hit by the incoming acoustic field u^i appears, outside a given set containing Ω and Ω_G, "as similar as possible" to the wave scattered in the same circumstances by the ghost obstacle $(\Omega_G;\chi_G)$.

Remember that a pressure current is a quantity whose physical dimension is: pressure divided by time.

Our goal is to model the ghost obstacle problem as an optimal control problem introducing a control variable $\psi(\underline{x},t), (\underline{x},t)\in\partial\Omega\times\mathbf{R},$ that is a pressure current acting on the boundary of the obstacle. To this aim, we replace the boundary condition (2) with the following boundary condition:

$$-\frac{\partial u^s}{\partial t}(\underline{x},t)+c\chi\frac{\partial u^s}{\partial \underline{n}(\underline{x})}(\underline{x},t)=g(\underline{x},t)+(1+\chi)\psi(\underline{x},t), (\underline{x},t)\in\partial\Omega\times\mathbf{R}. \tag{6}$$

Let Ω_ε be a bounded simply connected open set containing Ω and Ω_G with Lipschitz boundary $\partial\Omega_\varepsilon$ and let $ds_{\Omega_\varepsilon}, ds_{\partial\Omega}$ be the surface measures on $\partial\Omega_\varepsilon$ and $\partial\Omega$ respectively.

We choose the following cost functional:

$$F_{\lambda,\mu,\varepsilon}(\psi)=\int_{\mathbf{R}}dt\left\{\int_{\partial\Omega_\varepsilon}(1+\chi)\lambda\left(u^s(\underline{x},t)-u_G^s(\underline{x},t)\right)^2 ds_{\partial\Omega_\in}+\int_{\partial\Omega}(1+\chi)\mu\varsigma\psi^2(\underline{x},t)ds_{\partial\Omega}\right\}, \tag{7}$$

where $\lambda\geq 0,\mu\geq 0$ are adimensional constants such that $\lambda+\mu=1$, and ς is a nonzero positive dimensional constant. We model the ghost obstacle problem via the following optimal control problem:

$$\min_{\psi\in C}F_{\lambda,\mu,\varepsilon}(\psi), \tag{8}$$

subject to the constraints (1), (4), (5) and (6).

This is a legitimate mathematical model of the ghost obstacle problem. In fact the minimization $F_{\lambda,\mu,\varepsilon}$ makes small $u^s - u_G^s$ for $(\underline{x},t) \in \partial\Omega_\varepsilon \times \mathbf{R}$, that is makes small $u^s - u_G^s$ for $(\underline{x},t) \in (\mathbf{R}^3 \setminus \Omega_\varepsilon) \times \mathbf{R}$ and makes small the "size" of the pressure current used while the constraints (1), (4), (5), (6) guarantee the satisfaction of the dynamic conditions associated to the problem considered.

The set C is the space of the admissible controls that we leave undetermined. The obstacle $(\Omega;\chi)$ that generates the scattered field u^s solution of (8), (1), (4), (5), (6) is called smart or active obstacle.

Note that in (7) the choice $\Omega_G \subset \Omega, \Omega_\varepsilon = \Omega$ gives the masking problem and that the choice $\Omega_G = \phi$, $\Omega_\varepsilon = \Omega$ gives the furtivity problem.

3.0 THE FIRST ORDER OPTIMALITY CONDITIONS

Let us make the following assumptions: let (r,θ,ϕ) be the usual spherical coordinate system in $\mathbf{R}^3$ with pole in the origin, let B be the sphere with center the origin and radius one and let ∂B be its boundary, we assume that:

(a) the boundary of the obstacle Ω is a starlike surface with respect to the origin, that is Ω and ∂B can be represented as follows:

$$\Omega = \left\{ \underline{x} = r\underline{\hat{x}} \in \mathbf{R}^3 \middle| 0 \le r < \xi(\underline{\hat{x}}), \underline{\hat{x}} \in \partial B \right\}, \tag{9}$$

$$\partial\Omega = \left\{ \underline{x} = r\underline{\hat{x}} \in \mathbf{R}^3 \middle| r = \xi(\underline{\hat{x}}), \underline{\hat{x}} \in \partial B \right\}, \tag{10}$$

where $\xi(\underline{\hat{x}}) > 0, \underline{\hat{x}} \in \partial B$, is a single valued function defined on ∂B that is assumed sufficiently regular for the manipulations that follow;

(b) the sets Ω_ε and $\partial\Omega_\varepsilon$ can be represented as follows:

$$\Omega_\varepsilon = \left\{ \underline{x} = r\underline{\hat{x}} \in \mathbf{R}^3 \middle| 0 \le r < (\xi(\underline{\hat{x}}) + \varepsilon), \underline{\hat{x}} \in \partial B \right\}, \varepsilon > 0, \tag{11}$$

$$\partial\Omega_\varepsilon = \left\{ \underline{x} = r\underline{\hat{x}} \in \mathbf{R}^3 \middle| r = \xi(\underline{\hat{x}}) + \varepsilon, \underline{\hat{x}} \in \partial B \right\}, \varepsilon > 0. \tag{12}$$

for a suitable choice of $\varepsilon > 0$.

Note that the assumptions (*a*) and (*b*) are only one of many other possible choices of assumptions that can be made to guarantee the satisfactory solution of the model (8), (1), (4), (5), (6). This choice is made just to fix the ideas and to keep the exposition simple.

Under the assumptions (*a*) and (*b*), applying the Pontryagin maximum principle the optimal state trajectory $\tilde{u}^s$ and the corresponding adjoint variable trajectory $\tilde{\varphi}$ satisfy the necessary first order optimality conditions associated to the optimal control problem (8), (1), (4), (5), (6), that is they are the solution of the following exterior problem for a system of two coupled wave equations:

$$\Delta\tilde{u}^s(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2\tilde{u}^s}{\partial t^2}(\underline{x},t) = 0, \; (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \tag{13}$$

$$\tilde{u}^s(\underline{x},t) = O\left(\frac{1}{r}\right), \; r \to +\infty, \; t \in \mathbf{R}, \tag{14}$$

$$\frac{\partial \tilde{u}^s}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial \tilde{u}^s}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty,\ t\in\mathbf{R}, \tag{15}$$

$$\begin{aligned}-\frac{\partial \tilde{u}^s}{\partial r}(\underline{x},t)+c\chi\frac{\partial \tilde{u}^s}{\partial \underline{n}(\underline{x})}(\underline{x},t)&=g(\underline{x},t)-\\ -\frac{(1+\chi)}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t),&(\underline{x},t)\in\partial\Omega\times\mathbf{R},\end{aligned} \tag{16}$$

$$\Delta\tilde{\varphi}(\underline{x},t)-\frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial t^2}(\underline{x},t)=0,\ (\underline{x},t)\in\left(\mathbf{R}^3\setminus\overline{\Omega}\right)\times\mathbf{R}, \tag{17}$$

$$\tilde{\varphi}(\underline{x},t)=O\left(\frac{1}{r}\right),\ \ r\to+\infty,\ t\in\mathbf{R}, \tag{18}$$

$$\frac{\partial \tilde{\varphi}}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial \tilde{\varphi}}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty,\ t\in\mathbf{R}, \tag{19}$$

$$-\frac{\partial \tilde{\varphi}}{\partial t}(\underline{x},t)-c\chi\frac{\partial \tilde{\varphi}}{\partial \underline{n}(\underline{x})}(\underline{x})=\ -2\lambda(1+\chi)f_\varepsilon\left(\frac{\underline{x}}{\|\underline{x}\|}\right)\left(\tilde{u}^s\left(x+\varepsilon\frac{\underline{x}}{\|\underline{x}\|},t\right)-u_G^s\left(\underline{x}+\varepsilon\frac{\underline{x}}{\|\underline{x}\|},t\right)\right),(\underline{x},t)\in\partial\Omega\times\mathbf{R}, \tag{20}$$

$$\lim_{t\to-\infty}\tilde{u}^s(\underline{x},t)=0,\underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}, \tag{21}$$

$$\lim_{t\to+\infty}\tilde{\varphi}(\underline{x},t)=0,\underline{x}\in\mathbf{R}^3\setminus\Omega, \tag{22}$$

where $f_\varepsilon\left(\underline{x}/\|\underline{x}\|\right),x\in\partial\Omega$ is the function defined by:

$$\begin{aligned}&f_\varepsilon\left(\frac{\underline{x}}{\|\underline{x}\|}\right)=f_\varepsilon(\hat{\underline{x}}(\theta,\phi))=\frac{\upsilon_\varepsilon(\theta,\phi)}{\upsilon(\theta,\phi)},\underline{x}\in\partial\Omega,\hat{\underline{x}}=\frac{\underline{x}}{\|\underline{x}\|}\in\partial B,\\ &0\le\theta\le\pi,0\le\phi<2\pi,\end{aligned} \tag{23}$$

$$\begin{aligned}&\upsilon(\theta,\phi)=\xi\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^2\sin^2\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^2+\xi^2\sin^2\theta},\\ &0\le\theta\le\pi,0\le\phi<2\pi,\end{aligned} \tag{24}$$

$$\begin{aligned}&\upsilon_\varepsilon(\theta,\phi)=(\xi+\varepsilon)\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^2\sin^2\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^2+(\xi+\varepsilon)^2\sin^2\theta},\\ &0\le\theta\le\pi,0\le\phi<2\pi.\end{aligned} \tag{25}$$

The relation between $\tilde{\varphi}$ and the optimal control $\tilde{\psi}$ solution of problem (8), (1), (4), (5), (6) is the following one:

$$\tilde{\psi}(\underline{x},t)=-\frac{1}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R}. \tag{26}$$

Let us point out that we have:

$$ds_{\partial\Omega} = \upsilon(\theta,\phi)d\theta d\phi, 0 \le \theta \le \pi, 0 \le \phi < 2\pi\,, \quad (27)$$

and

$$ds_{\partial\Omega\varepsilon} = \upsilon_{\varepsilon}(\theta,\phi)d\theta d\phi, 0 \le \theta \le \pi, 0 \le \phi < 2\pi\,, \quad (28)$$

4.0 NUMERICAL SOLUTION OF THE EXTERIOR PROBLEM (13) – (22)

Numerical methods to solve the exterior problem (13)-(22) have been developed in [5], [6]. These methods belong to the class of the operator expansion methods and are highly parallelizable. Some numerical experiments proving the validity of the control problem proposed as mathematical model of the ghost obstacle problem are shown in the website http://www.econ.univpm.it/recchioni/w11.

5.0 EXTENSION AND CONCLUSIONS

The work presented can be extended to a new class of smart obstacles that pursue the following goal:

4. one of the goals specified in the Introduction restricted to a definite band in the frequency space.

For reasons of brevity we restrict our attention to the definite band ghost obstacle problem in the acoustic case. This problem is formulated as an optimal control problem for the wave equation. The acoustic definite band masking problem and furtivity problem can be treated similarly. We consider the study of these problems as preliminary to the study of the corresponding problems in the electromagnetic case where the wave equation must be replaced with the Maxwell equations.

We note that restricting the goal pursued to a definite band in the frequency space modifies substantially the mathematical formulation of the problems under scrutiny. In fact the optimal control problems used to model problems 1), 2), 3) in particular the cost functionals that must be minimized in order to model appropriately the problem formulated only on the desired band in the frequency space are "nonlocal". That is the presence of the definite band makes necessary the use of suitable convolutions involving the anti Fourier transform of the characteristic function of the definite frequency band in the definition of the cost functional. Consequently the first order optimality conditions of these new optimal control problems change substantially and cannot be deduced from those derived in [2], [3], [5] and here for the optimal control problems 1), 2), 3). That is the first order optimally conditions are not expressed by two wave equations coupled by local boundary conditions as in [2], [3], [5] and here but the coupling between the two wave equations is given by nonlocal (in time) boundary conditions. As a consequence the way of solving the first order optimality conditions must be changed.

We can conclude that the idea of modelling the smart obstacles using optimal control problems is an interesting idea. Moreover the work developed until now with the model proposed can be profitably extended in several directions such as the study of closed loop controls, finite horizon controls, or the study of inverse problems involving smart obstacles. These are challenging mathematical questions whose solution can be very valuable in practical applications.

6.0 REFERENCES

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